

Static Optimization with MATLAB

Day 1: Unconstrained Optimization

GSEFM Pre-Semester Course

Luca Schmitz

Institute for Monetary and Financial Stability, Goethe University Frankfurt

September 22-28, 2026 | Frankfurt

Content

1. Theory

- Necessary Conditions
- Sufficient Conditions
- Concave Optimization

2. Exercises

3. MATLAB Implementation

Optimization Problem I

Definition (Static Optimization Problem)

An optimization problem consists of a set of feasible solutions M and an objective function $f : M \rightarrow \mathbb{R}$. The goal is to find the maximum value of $f(x)$ over all $x \in M$, which is expressed as:

$$\max_{x \in M} f(x)$$

- If $M = \mathbb{R}^n$, we speak of an **unconstrained optimization problem**.

Optimization Problem II

Definition (Solution)

- $x^* \in M$ is called a global maximum of the optimization problem if $f(x^*) \geq f(x)$ for all $x \in M$.
- $x^* \in M$ is called a strict global maximum of the optimization problem if $f(x^*) > f(x)$ for all $x \in M$.
- $x^* \in M$ is called a local maximum if there exists a neighborhood U^* of x^* such that $f(x^*) \geq f(x)$ for all $x \in M \cap U^*$.
- $x^* \in M$ is called a strict local maximum if there exists a neighborhood U^* of x^* such that $f(x^*) > f(x)$ for all $x \in M \cap U^*$.

Necessary Conditions I

Theorem (First Order Necessary Condition)

Let $M \subseteq \mathbb{R}^n$ an open set and let $f : M \rightarrow \mathbb{R}$ be a continuously differentiable function. Let x^ be a local maximum of f (on M), then*

$$\nabla f(x^*) = 0,$$

i.e., x^ is a so-called stationary point of f .*

This condition is not sufficient, e.g. $f(x) = x^3$ has a stationary point at $x^* = 0$, but x^* is not a local maximum.

Necessary Conditions II

Theorem (Second Order Necessary Condition)

Let $M \subseteq \mathbb{R}^n$ an open set and let $f : M \rightarrow \mathbb{R}$ be twice continuously differentiable. Let x^ be a local maximum of f (on M), then*

$$\nabla^2 f(x^*) \preceq 0,$$

i.e., the Hessian of f at x^ is negative semi-definite* Definition.

This condition is not sufficient, e.g. $n = 2$, $f(x) = -x_1^2 + x_2^4$, $x^* = (0, 0)$ has negative semi-definite Hessian, but x^* is not a local maximum.

Sufficient Conditions

Theorem (Sufficient Condition)

Let $M \subseteq \mathbb{R}^n$ an open set and let $f : M \rightarrow \mathbb{R}$ be twice continuously differentiable. If

- $\nabla f(x^*) = 0$, and
- $\nabla^2 f(x^*) \prec 0$,

then x^* is a strict local maximum of f (on M). If $\nabla^2 f(x^*) \succ 0$, we have a strict local minimum. And if $\nabla^2 f(x^*)$ is indefinite, we have a saddle point.

This condition is not necessary, e.g. $n = 2$, $f(x) = -x_1^2 - x_2^4$, $x^* = (0, 0)$ has negative **semi**-definite Hessian, but x^* is also a strict local maximum.

Why M is required to be open?

- In the unconstrained case, $M = \mathbb{R}^n$ is open (and closed).
- Technically, e.g. the case of a closed interval, $M = [a, b]$, $a < b$, $a, b, \in \mathbb{R}$, is already a linear constraint:

$$g(x) := \begin{pmatrix} a - x \\ x - b \end{pmatrix} \leq 0$$

► Dealing with boundary points

- In the case of an open interval, $M = (a, b)$, $a < b$, $a, b, \in \mathbb{R}$, we have a strict linear constraint, which is usually ill-posed, as we need our problem to be posed on a compact domain to ensure that our (continuous) objective function obtains its extremum. ► Weierstrass Theorem ► Example

Concavity

Theorem (Concavity Characterizations)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function over an open domain. Then the following statements are equivalent:

- *f is concave, i.e., for all $\lambda \in [0, 1]$ and all $x, y \in \mathbb{R}^n$, it holds that*

$$f((1 - \lambda)x + \lambda y) \geq (1 - \lambda)f(x) + \lambda f(y)$$

- *for all $x, y \in \mathbb{R}^n$, it holds that, $f(y) \leq f(x) + \nabla f(x)(y - x)$*
- *for all $x \in \mathbb{R}^n$, it holds that $\nabla^2 f(x) \preceq 0$.*

Concave Optimization

Theorem

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ continuously differentiable and concave, and let x^ be a stationary point of f . Then x^* is a global maximum of f .*

Extension to Generalized Concave Problems I

Definition (Pseudoconcave Function)

A continuously differentiable function f is pseudoconcave in a set M if, given any two points x and y in X , we have that

$$Df(x)(y - x) \leq 0 \Rightarrow f(y) \leq f(x)$$

concave $\Rightarrow$ pseudoconcave $\Rightarrow$ strictly quasiconcave $\Rightarrow$ quasiconcave (for differentiable functions)

Extension to Generalized Concave Problems II

Intuition

- In optimization (maximization) concave objective function f are very useful, because from a stationary point $x^* \in \mathbb{R}^n$, i.e., $\nabla f(x^*) = 0$, it follows directly that x^* is a (global=local) maximum, i.e., $f(x^*) \geq f(x)$ for all $x \in \mathbb{R}^n$.
 - A pseudoconcave function tries to mimic this property. It is not as strong as convexity and therefore applies to a broader range of objective functions:
- ⇒ Given a stationary point x^* , it holds that

$$0 = \nabla f(x^*)(y - x^*) \leq 0 \quad \forall y \in \mathbb{R}^n,$$

and thus (by the definition of a pseudoconcave function), $f(y) \leq f(x^*)$ for all $y \in \mathbb{R}^n$, i.e., x^* is a (global=local) maximum.

Extension to Generalized Concave Problems III

Theorem

Let f be a continuously differentiable and pseudoconcave ► Definition function. If $x^ \in M$, M convex, and for every direction $h \in \mathbb{R}^n$ feasible from x^* we have $Df(x^*)h \leq 0$, then x^* is an optimal solution of the maximization problem.*

Extension to Generalized Concave Problems IV

Definition (Strictly Quasiconcave Function)

Let f be a real-valued function defined on a convex set M . f is strictly quasiconcave if for all x and y in M and all $\lambda \in (0, 1)$ we have

$$f((1 - \lambda)x + \lambda y) > \min\{f(x), f(y)\}$$

concave $\Rightarrow$ pseudoconcave $\Rightarrow$ strictly quasiconcave $\Rightarrow$ quasiconcave (for differentiable functions)

Extension to Generalized Concave Problems V

Theorem

Let x^* be an optimal solution to the maximization problem, with M convex. If f is **strictly quasiconcave** ► Definition, then x^* is the unique optimal solution to the problem.

Exercise 1

Consider a competitive firm that produces a single output y using two inputs x_1 and x_2 . The firm's production technology is described by a Cobb-Douglas function

$$y = f(x_1, x_2) = x_1^\alpha x_2^\beta, \quad \text{where } \alpha + \beta < 1, \beta > 0, \alpha > 0$$

Taking as given the price of its output, p , and input prices w_1 and w_2 , the firm maximizes its profits given by

$$\Pi(x_1, x_2) = px_1^\alpha x_2^\beta - w_1 x_1 - w_2 x_2$$

1. Write the first-order conditions for the firm's problem
2. Check that sufficient conditions for a maximum are satisfied
3. Solve for the firm's factor demands $x_i^* = x_i(p, w_1, w_2)$, for $i = \{1, 2\}$

Exercise 1 Solution I

1. FOCs:

$$\frac{\partial \Pi}{\partial x_1}(x_1^*, x_2^*) = p\alpha x_1^{*\alpha-1} x_2^{*\beta} - w_1 = 0$$

$$\frac{\partial \Pi}{\partial x_2}(x_1^*, x_2^*) = p\beta x_1^{*\alpha} x_2^{*\beta-1} - w_2 = 0$$

Exercise 1 Solution II

2. Sufficient conditions for a maximum: We have to show that the production function $f()$ is strictly concave $\Rightarrow$ i.e. that the Hessian is negative definite.

$$f_1 = \alpha x_1^{\alpha-1} x_2^\beta = \frac{\alpha y}{x_1}, \quad f_2 = \beta x_1^\alpha x_2^{\beta-1} = \frac{\beta y}{x_2},$$

$$f_{11} = \alpha(\alpha-1)x_1^{\alpha-2}x_2^\beta = \frac{\alpha(\alpha-1)y}{x_1^2},$$

$$f_{22} = \beta(\beta-1)x_1^\alpha x_2^{\beta-2} = \frac{\beta(\beta-1)y}{x_2^2},$$

$$f_{12} = f_{21} = \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} = \frac{\alpha\beta y}{x_1 x_2}$$

Exercise 1 Solution III

$$H = \begin{bmatrix} \frac{\alpha(\alpha-1)y}{x_1^2} & \frac{\alpha\beta y}{x_1 x_2} \\ \frac{\alpha\beta y}{x_1 x_2} & \frac{\beta(\beta-1)y}{x_2^2} \end{bmatrix}$$

- $|H_1| = \frac{\alpha(\alpha-1)y}{x_1^2} < 0$

- $|H_2| = \frac{\alpha(\alpha-1)y}{x_1^2} \frac{\beta(\beta-1)y}{x_2^2} - \left(\frac{\alpha\beta y}{x_1 x_2} \right)^2 = \frac{\alpha\beta[1-\alpha-\beta]y^2}{x_1^2 x_2^2} > 0$

$\Rightarrow$ Conditions fulfilled $\Rightarrow f()$ is concave $\Rightarrow \Pi()$ is thus also concave (because costs are linear $\Rightarrow$ concave $\Rightarrow$ sum of concave functions is also concave)

Exercise 1 Solution IV

3. Input demand functions.

From the FOCs we have

$$\frac{p\alpha y}{x_1^*} = w_1, \quad \frac{p\beta y}{x_2^*} = w_2$$

Dividing the two by each other:

$$\frac{w_1}{w_2} = \frac{\alpha x_2^*}{\beta x_1^*} \Leftrightarrow x_2^* = \frac{w_1 \beta}{w_2 \alpha} x_1^*$$

Plug into first FOC:

$$\begin{aligned} w_1 &= p\alpha x_1^{*\alpha-1} \left(\frac{w_1 \beta}{w_2 \alpha} x_1^* \right)^\beta \\ \Rightarrow x_1^* &= x_1(p, w_1, x_2) = \left(\frac{p\alpha^{1-\beta} w_1^{\beta-1} \beta^\beta}{w_2^\beta} \right)^{\frac{1}{1-\alpha-\beta}} \end{aligned}$$

Can then also solve for x_2^* .

Exercises 2

Find the critical points of

1. $f(x_1, x_2) = x_1^3 - x_2^3 + 9x_1x_2, \quad x_1, x_2 \in \mathbb{R}^2$
2. $f(x_1, x_2) = -x_1 \ln(x_1) - x_2 \ln(x_2), \quad x_1, x_2 \in \mathbb{R}^2$
3. $f(x_1, x_2) = x_2 - 4x_1^2 + 3x_1x_2 - x_2^2, \quad x_1, x_2 \in \mathbb{R}^2$

and check whether it is a local maximum, minimum, or saddle point.

Solution Exercise 2.1 I

First-order conditions (Jacobian):

$$\nabla f(x_1^*, x_2^*) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x_1^*, x_2^*) \\ \frac{\partial f}{\partial x_2}(x_1^*, x_2^*) \end{pmatrix} = \begin{pmatrix} 3x_1^{*2} + 9x_2^* \\ -3x_2^{*2} + 9x_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving the system of equations yields critical points:

$$\begin{aligned} (x_{1,1}^*, x_{2,1}^*) &= (0, 0) & \Rightarrow f(x_{1,1}^*, x_{2,1}^*) &= 0 \\ (x_{1,2}^*, x_{2,2}^*) &= (3, -3) & \Rightarrow f(x_{1,2}^*, x_{2,2}^*) &= -27 \end{aligned}$$

Second-order conditions:

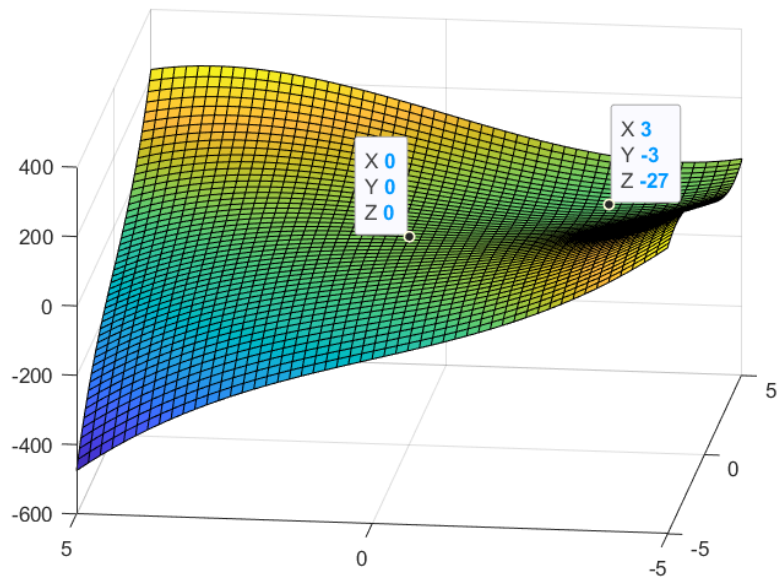
Solution Exercise 2.1 II

$$H(x_1^*, x_2^*) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x_1^*, x_2^*) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1^*, x_2^*) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1^*, x_2^*) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x_1^*, x_2^*) \end{pmatrix} = \begin{pmatrix} 6x_1^* & 9 \\ 9 & -6x_2^* \end{pmatrix}$$

Checking the two critical points:

1. $(x_{1,1}^*, x_{2,1}^*) = (0, 0)$: $H(0, 0) = \begin{pmatrix} 0 & 9 \\ 9 & 0 \end{pmatrix}$
 $\Rightarrow |H_1| = 0, |H_2| = -81 \Rightarrow \text{Inconclusive}$
2. $(x_{1,2}^*, x_{2,2}^*) = (3, -3)$: $H(3, -3) = \begin{pmatrix} 18 & 9 \\ 9 & 18 \end{pmatrix}$
 $\Rightarrow |H_1| = 18, |H_2| = 243 \Rightarrow \text{Local minimum}$

Solution Exercise 2.1 III



Solution Exercise 2.2 I

First-order conditions (Jacobian):

$$\nabla f(x_1^*, x_2^*) = \begin{pmatrix} -\ln(x_1^*) - 1 \\ -\ln(x_2^*) - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving the system of equations yields the critical point:

$$(x_1^*, x_2^*) = (e^{-1}, e^{-1}) \Rightarrow f(x_1^*, x_2^*) = 2e^{-1}$$

Second-order condition:

$$H(x_1^*, x_2^*) = \begin{pmatrix} -\frac{1}{x_1^*} & 0 \\ 0 & -\frac{1}{x_2^*} \end{pmatrix}$$

Checking the critical point:

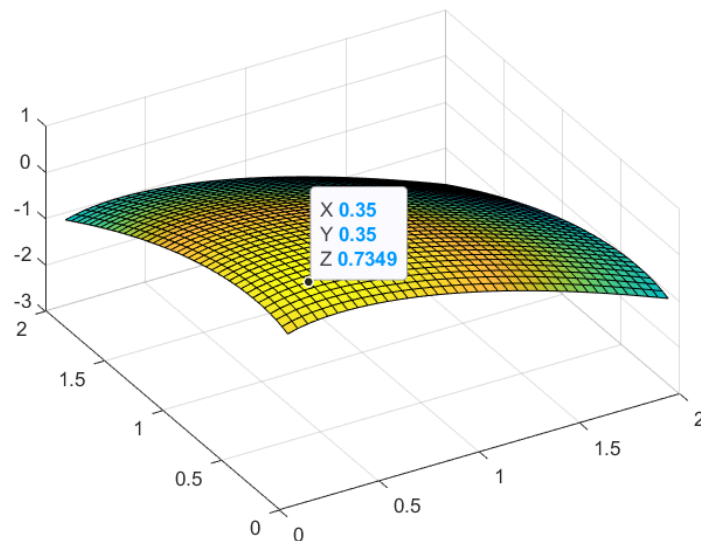
Solution Exercise 2.2 II

$$(x_1^*, x_2^*) = (e^{-1}, e^{-1}): H(e^{-1}, e^{-1}) = \begin{pmatrix} -\frac{1}{e^{-1}} & 0 \\ 0 & -\frac{1}{e^{-1}} \end{pmatrix}$$

$$\Rightarrow |H_1| \approx -2.7183, |H_2| \approx 7.3891$$

$\Rightarrow$ local maximum point

Solution Exercise 2.2 III



Solution Exercise 2.3 I

First-order conditions:

$$\nabla f(x_1^*, x_2^*) = \begin{pmatrix} -8x_1 + 3x_2 \\ 1 + 3x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving the system of equations yields the critical point:

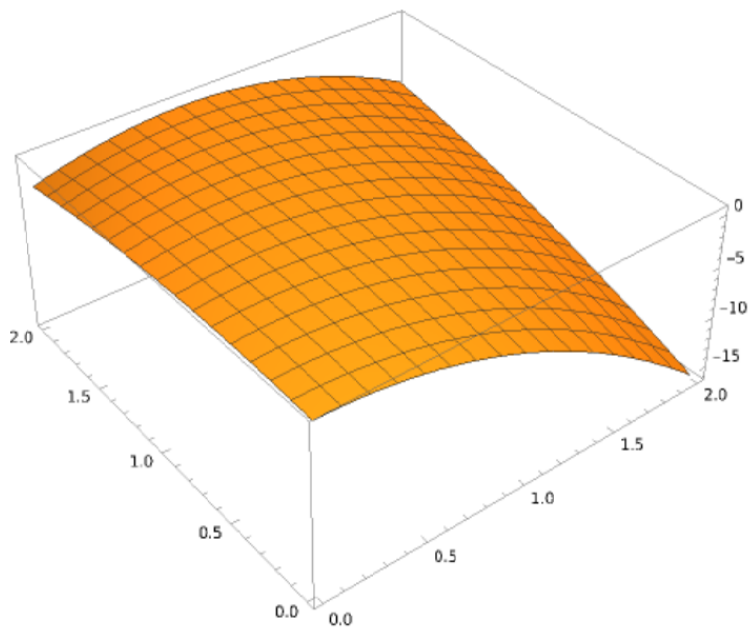
$$(x_1^*, x_2^*) = (3/7, 8/7) \Rightarrow f(x_1^*, x_2^*) = 4/7$$

Second-order condition:

$$H(x_1^*, x_2^*) = \begin{pmatrix} -8 & 3 \\ 3 & -2 \end{pmatrix}$$

$$\Rightarrow |H_1| = -8, |H_2| = 7 \Rightarrow \text{local maximum point}$$

Solution Exercise 2.3 II



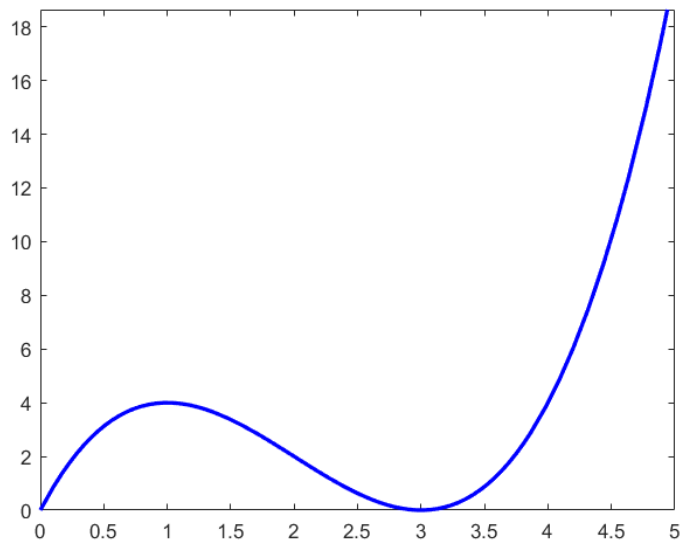
References I

- Alt, W. (2011). *Nichtlineare optimierung eine einführung in theorie, verfahren und anwendungen: Eine einführung in theorie, verfahren und anwendungen*. Vieweg+Teubner Verlag.
- Chiang, A. C., & Wainwright, K. (2005). *Fundamental methods of mathematical economics*. McGraw-Hill Education.
- de la Fuente, A. (2000). *Mathematical methods and models for economists*. Cambridge University Press.
- Geiger, C., & Kanzow, C. (1999). *Numerische verfahren zur lösung unrestringierter optimierungsaufgaben (springer-lehrbuch)*. Springer.
- Geiger, C., & Kanzow, C. (2002). *Theorie und numerik restringierter optimierungsaufgaben*. Springer.
- Ulbrich, M. (2011). *Nichtlineare optimierung*. Birkhäuser Verlag.
- Werner, J. (2002). Unrestringierte optimierungsaufgaben [Last visited on 2023/09/19]. <http://num.math.uni-goettingen.de/werner/uncopt.pdf>
- Werner, J. (2012). Vorlesung über optimierung [Last visited on 2023/09/19]. <https://num.math.uni-goettingen.de/werner/optim.pdf>

Dealing with boundary points I

Let $f : [0, 5] \rightarrow \mathbb{R}$ such that $f(x) = x^3 - 6x^2 + 9x$.
Find the local and global extrema.

Dealing with boundary points II



Dealing with boundary points III

Consider the problem

$$\max_{x \in S} f(x),$$

where S is a convex set.

Theorem

Assume that f is continuously differentiable and let x^ be an optimal solution of the maximization problem, then it holds that*

$$\lim_{\alpha \rightarrow 0} \frac{f(x^* + \alpha h) - f(x^*)}{\alpha} = Df(x^*)h \leq 0,$$

for every direction vector $h \in \mathbb{R}^n$ feasible from x^ .*

Dealing with boundary points IV

Definition

We say that h is a feasible direction from x if there exists some $\delta > 0$ such that

$$x + \alpha h \in \mathcal{S} \quad \forall \alpha \in (0, \delta).$$

Idea: A zero derivative is neither necessary nor sufficient for a maximum (without further assumptions).

Dealing with boundary points V

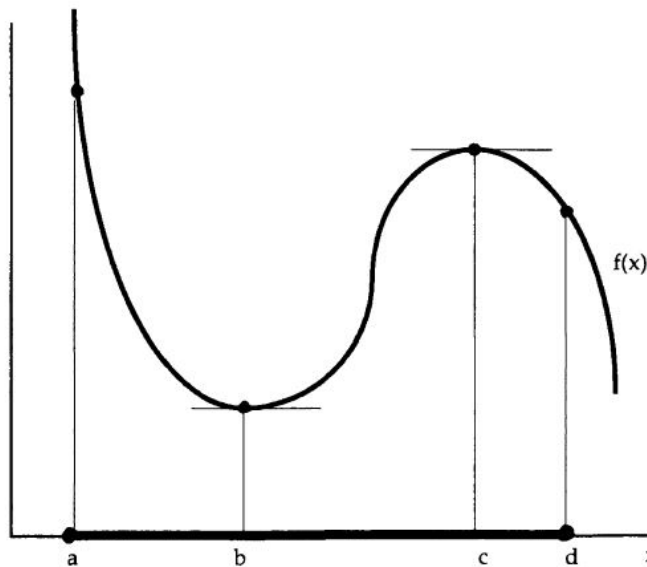


Figure 7.1. A zero derivative is neither necessary nor sufficient for a maximum.

Dealing with boundary points VI

Recall:

$$\max_{x \in [0,5]} f(x) = x^3 - 6x^2 + 9x = x(x-3)^2$$

Check endpoints:

- $x = 0$: Only one feasible direction: $h > 0$:

$$\lim_{\alpha \rightarrow 0} \frac{(0 + \alpha h)((0 + \alpha h) - 3)^2 - 0}{\alpha} = \lim_{\alpha \rightarrow 0} h(\alpha h - 3)^2 > 0$$

$\Rightarrow$ necessary condition not fulfilled

Dealing with boundary points VII

- $x = 5$: Only one feasible direction. $h < 0$:

$$\lim_{\alpha \rightarrow 0} \frac{(5 + \alpha h)((5 + \alpha h) - 3)^2 - 5(5 - 3)^2}{\alpha}$$
$$\lim_{\alpha \rightarrow 0} \frac{5 \underbrace{[(5 + \alpha h) - 3]^2 - (5 - 3)^2}_{< 0} + \underbrace{\alpha h}_{< 0} ((5 + \alpha h) - 3)^2}{\alpha} < 0$$

$\Rightarrow$ necessary condition fulfilled

Dealing with boundary points VIII

Interior points:

- First set $Df(x) = 0$, i.e. check at which point the derivative of the function is zero

$$Df(x) = 3x^2 - 12x + 9 = 0 \Rightarrow x^* = 1, 3$$

- Then proceed as above, by checking directional derivatives around these points (both-sided)

$\Rightarrow$ will get that $Df(x^* = 1)h < 0$, but $Df(x^* = 3)h > 0$, thus only $x^* = 1$ is an additional local maximum.

Dealing with boundary points IX

- If the constraint set is an **open** set, all points are by definition interior, and given any $x \in \mathcal{S}$, all directions are feasible from it.
- ⇒ In this case, the inequality $Df(x^*)h \leq 0$, $\forall h$ can only hold if all first partial derivatives of f are zero at x^* , i.e., for each $i = 1, \dots, n$ we have

$$\frac{\partial f(x)}{\partial x_i} = 0,$$

- ⇒ Openness guarantees only interior solutions. There are no boundary points we have to take care of.
- For the above problem, this would be the case if $x \in (0, 5)$.

► Go Back

Weierstrass Theorem

Theorem (Weierstrass Theorem)

A continuous, real-valued function $f : M \rightarrow \mathbb{R}$ defined on a compact set $M \subseteq \mathbb{R}^n$ achieves a (global) minimum and a (global) maximum in M , i.e. $\exists a, b$ such that:

$$f(b) \geq f(x) \geq f(a), \quad \forall x \in M$$

► Go Back

Optimization with strict inequality

Consider

$$\min_{x \in (0,1)} x$$

This problem doesn't have an optimal solution, since, given any $x > 0$, $x/2$ produces a solution with strictly better objective value. [▶ Go Back](#)

Pseudoconcave Function I

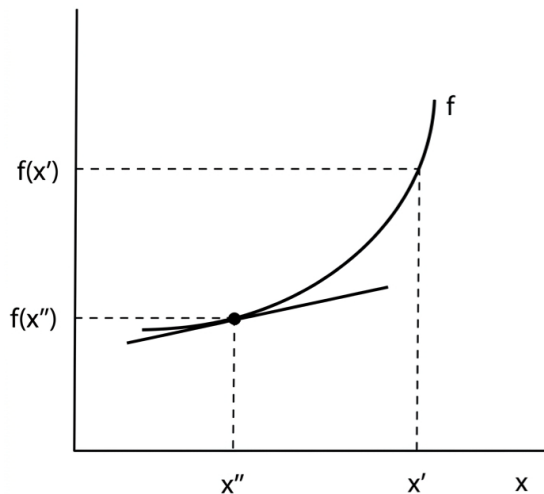
Definition (Pseudoconcave Function)

A continuously differentiable function f is pseudoconcave in a set M if, given any two points x and y in X , we have that

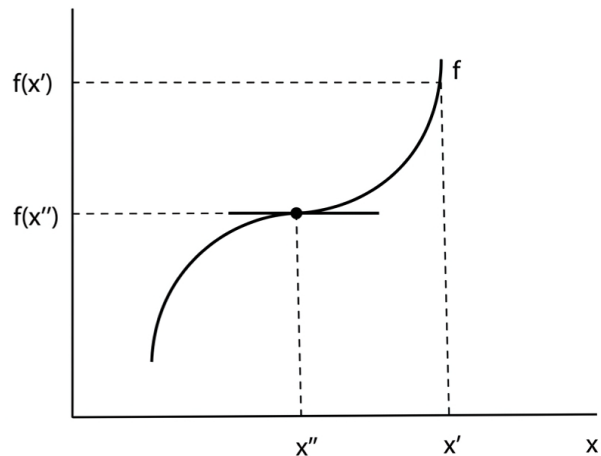
$$f(x) > f(y) \Rightarrow Df(y)(x - y) > 0$$

concave $\Rightarrow$ pseudoconcave $\Rightarrow$ strictly quasiconcave $\Rightarrow$ quasiconcave (for differentiable functions)

Pseudoconcave Function II



(a) f is strictly quasiconcave $Df(x'')(x' - x'') > 0$



(b) f is quasiconcave, but not strictly $Df(x'')(x' - x'') = 0$

Figure: Pseudoconcave function

Strictly Quasiconcave Function

Definition (Strictly Quasiconcave Function)

Let f be a real-valued function defined on a convex set M . f is strictly quasiconcave if for all x and y in M and all $\lambda \in (0, 1)$ we have

$$f((1 - \lambda)x + \lambda y) > \min\{f(x), f(y)\}$$

concave $\Rightarrow$ pseudoconcave $\Rightarrow$ strictly quasiconcave $\Rightarrow$ quasiconcave (for differentiable functions) [▶ Go Back](#)

Definition Positive/Negative Definite Matrix I

Theorem

Let A be a $n \times n$ symmetric matrix, and let A_k be the submatrix of A obtained by taking the upper-left-hand corner $k \times k$ submatrix of A . Furthermore, let $\Delta_k = |A_k|$, the k th principle minor of A . Then, it holds that

- *A is positive (semi-)definite $\Leftrightarrow \det(A)_k(\geq) > 0$ for all leading principal minors,*
- *A is negative (semi-)definite $\Leftrightarrow (-1)^k \det(A)_k(\geq) > 0$ for all leading principal minors,*

Definition Positive/Negative Definite Matrix II

Exercise: Is the matrix negative definite?

$$\begin{pmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

▶ go back